Lyapunov stability:

- Stability is an important concept in engineering
- Lyapunov function method is a powerful tool to ensure stability and study convergence of solutions to eqlb. points
- Consider time-invariant (autonomous) systems

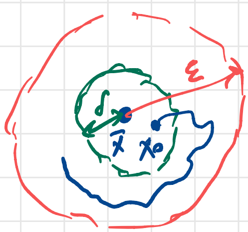
$$\dot{x} = f(x)$$

- Suppose $\bar{x}$ is an eqlb. point.
- Without loss of generality, we assume $\bar{x} = 0$

Def: $\bar{x}=0$ is stable if, $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon$$

Solution remains arbitrary close to $\bar{x}$ if it starts close enough to $\bar{x}$



Def: $\bar{x}=0$ is asymptotically stable (AS), if

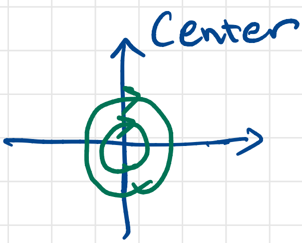
(1) it is stable

(2) and $\exists\ \delta_2>0$ s.t.

$$\|x(0)\|<\delta_2 \implies \lim_{t\to\infty} x(t)=0$$

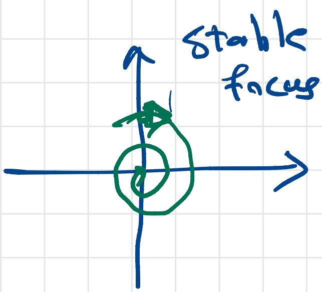
Solutions starting close enough to $\bar{x}=0$ would converge to $\bar{x}=0$

Example:



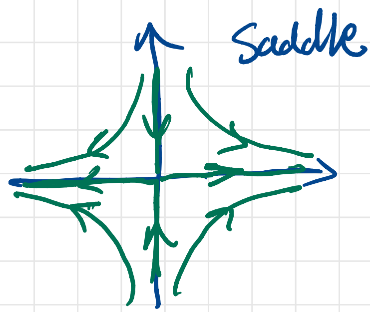
$\dot{x}_1=x_2$
$\dot{x}_2=-x_1$

stable, but not asymptotically stable



$\dot{x}_1=x_2$
$\dot{x}_2=-x_1-x_2$

asymptotically stable.



$\dot{x}_1=x_2$
$\dot{x}_2=x_1$

unstable

Def: The set of all initial conditions from which the solution converges to 0 is called region of attraction.

- For AS, region of attraction is a ball of radius δ_2

Def: $\bar{x}=0$ is Globally Asymptotically Stable (GAS) if it is AS and region of attraction $= \mathbb{R}^n$

- GAS $\Rightarrow$ AS

- These definitions do not consider the rate of convergence.

did not cover in class

Def: $\bar{x}=0$ is exponentially stable if $\exists\ \delta, C, \lambda > 0$ s.t.

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq C e^{-\lambda t} \|x(0)\| \quad \forall t \geq 0$$

Def: Globally exp. stable if every δ works.

Lyapunov functions:

- Let $V: \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable

- $V(x)$ is positive definite if

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \neq 0$$

- $V(x)$ is radially unbounded if

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty$$

Examples:

- $V(x) = x^T P x$ is positive-def and radially unbounded if P is positive-definite matrix

- $V(x) = (x_1 - x_2)^2$ is neither p.d nor radially unbounded

- Consider the system $\dot{x} = f(x)$

- $V(x(t))$ is the value of function V along the trajectory.

- The rate of change of $V(x(t))$ with time t

$$\frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x}(x(t))^T \dot{x}(t)$$

$$= \frac{\partial V}{\partial x}(x(t))^T f(x(t))$$

$$= \left[\frac{\partial V}{\partial x_1}(x(t)), \dots, \frac{\partial V}{\partial x_n}(x(t))\right] \begin{bmatrix} f_1(x(t)) \\ \vdots \\ f_n(x(t)) \end{bmatrix}$$

- One can disregard t and define

$$\dot{V}(x) \triangleq \frac{\partial V}{\partial x}(x)^T f(x) \quad \text{as function of } x$$

- This gives the rate of change of V for trajectory that passes x.

Example:

$$\dot{x}_{(t)} = a x_{(t)} \qquad x \in \mathbb{R}$$

$$V(x) = \frac{1}{2} x^2$$

then

$$\frac{d}{dt} V(x(t)) = \frac{\partial V}{\partial x}(x_{(t)})\, \dot{x}_{(t)}$$

$$= x(t) \quad a\, x(t)$$

$$= a\, x^2(t)$$

$$\Rightarrow \quad \dot{V}(x) = a x^2$$

Example:

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$$V(x) = \frac{1}{2} x^T P x$$

- A is $n \times n$ matrix
- P is $n \times n$ symmetric matrix.

$$\frac{\partial V}{\partial x}(x) = Px$$

(Review derivative of vector valued func.)

$$\Rightarrow \quad \dot{V}(x) = \frac{\partial V}{\partial x}^T(x)\, f(x) = x^T P A x$$

$$= x^T A^T P x$$

$$= \frac{1}{2} x^T (PA + A^T P) x$$

Thm (Thm. 4.1 and 4.2 in Khalil)

- Assume $\bar{x}=0$ is eqlb. of $\dot{x}=f(x)$
- Let $V:\mathbb{R}^n \to \mathbb{R}$ be positive def

① $\dot{V}(x) \leq 0 \quad \forall x \in D \to$ open set containing 0

$\Rightarrow \bar{x}=0$ is stable

② $\dot{V}(x) < 0 \quad \forall x \in D$
$x \neq 0$

$\Rightarrow \bar{x}=0$ is AS

③ $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n$ and
V is radially unbounded

$\Rightarrow \bar{x}=0$ is GAS

Proof:

① $\forall \varepsilon > 0$, we need to find δ s.t.

$$\text{if } \|X(0)\| < \delta \Rightarrow \|X(t)\| < \varepsilon \quad \forall t$$

- Pick $r \in (0, \varepsilon)$ s.t.

$$B_r = \{x \mid \|x\| \leq r\} \subset D$$

- Enough to show $\|X(t)\| \in B_r$

- Let $\alpha = \min_{\|x\| = r} V(x)$

- Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

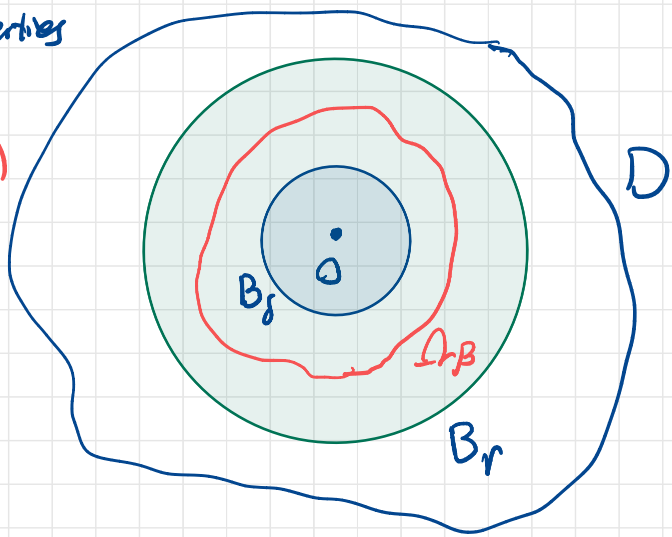
Ω_β has following properties

(i) $0 \in \Omega_\beta$

(ii) $\Omega_\beta \subset \text{int}(B_r)$ (**)

(iii) if $X(0) \in \Omega_\beta$

$\Rightarrow X(t) \in \Omega_\beta$ (*)



- why (iii) hold?

$$\dot{V}(X(t)) \leq 0 \implies V(X(t)) \leq V(X(0)) \leq \beta \quad \forall t \geq 0$$

- Therefore, we need to choose the initial condition inside the set Ω_β
- In other words, δ should be small enough s.t. $B_\delta \subset \Omega_\beta$

- Because $V(x)$ is continuous, and $V(0)=0$

$\implies \exists \delta$ s.t. if $\|X-0\| < \delta \overset{(**)}{\implies} V(x) < \beta$

- Therefore we have $B_\delta \subset \Omega_\beta \subset B_r$

$$X(0) \in B_\delta \overset{(**)}{\implies} X(0) \in \Omega_\beta \overset{(*)}{\implies} X(t) \in \Omega_\beta$$

$$\overset{(***)}{\implies} X(t) \in B_r \quad \forall t$$

$$\implies \|X(0)\| \leq \delta \implies \|X(t)\| \leq r < \varepsilon \quad \forall t$$

$\implies \bar{X} = 0$ is stable ∎

② assume $\dot{V}(x) < 0$

- need to show $X(t) \to 0$ as $t \to \infty$

- From part ① we know that $\|X(t)\| \leq \varepsilon \ \forall t$
- Moreover, $V(X(t))$ is decreasing in t and bounded from below $(V(x) \geq 0)$

- Therefore, $V(X(t))$ has a limit as $t \to \infty$

$$\lim_{t \to \infty} V(X(t)) = c$$

- If $c = 0$, we are done because $V(x) = 0$ only at $x = 0$

- If $c > 0$, we prove by contradiction

assume $V(X(t)) \geq c > 0$

$\exists d > 0$ s.t. if $\|x\| < d \Rightarrow V(x) < c$

$\Rightarrow \|X(t)\| \geq d$

- Let $\quad -\gamma = \max\limits_{d \le \|x\| \le r} \dot{V}(x)$

- By assumption $\dot{V}(x) < 0$ we know $\gamma > 0$

- $$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(t))\, dt$$

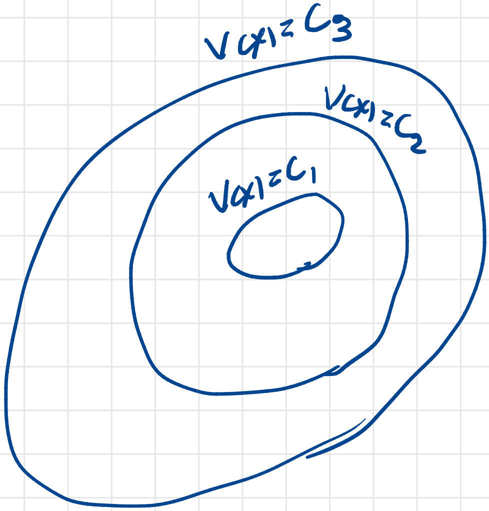
$$\le V(x(0)) - \gamma t$$

- The RHS becomes negative as $t \uparrow \infty$
- This contradicts the assumption that
$$V(x(t)) \geqslant 0 \implies c = 0$$

Pictorial argument:

$$\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \le c\}$$

level surface



- $\dot{V} \le 0 \Rightarrow$ stays inside
- $\dot{V} < 0 \Rightarrow$ has to go to zero